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Small-amplitude solitary structures for an extended nonlinear Schrödinger equation

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Abstract. A perturbative approach is used to obtain small-amplitude solitary structures for an extended nonlinear Schrödinger equation. These structures have the form of dark and anti-dark solitary wave solutions, closely connected with the Korteweg–deVries solitons. The solutions found are valid in wavelength regions, such as those applicable in the anomalous dispersion regime, which are not accessible by the conventional nonlinear Schrödinger equation. The dynamics of the derived structures in the presence of the Raman effect is also studied by means of a Korteweg–deVries–Burgers equation. The obtained results are applied to the problem of propagation of femtosecond duration pulses in nonlinear optical fibres.

1. Introduction

In the present paper we study solitary structures of an extended nonlinear Schrödinger (ENLS) equation, of the following form:

$$\mathrm{i}\frac{\partial q}{\partial \zeta} - \frac{s}{2}\frac{\partial^2 q}{\partial \tau^2} + q|q|^2 - \mathrm{i}\lambda\frac{\partial^3 q}{\partial \tau^3} + \mathrm{i}\mu\frac{\partial}{\partial \tau}(q|q|^2) + \mathrm{i}\nu q\frac{\partial}{\partial \tau}(|q|^2) = 0. \qquad (1)$$

Equation (1) has important applications in nonlinear optics [1], where it has been used to describe femtosecond pulse propagation in nonlinear optical fibres, up to distances much less than the absorption length [2–5]. The parameter s is the sign of the group velocity dispersion (GVD), while λ, μ and ν are positive real constants. This equation is not in general integrable by means of the inverse scattering transform (IST) [6]. Nevertheless, it can be transformed, under certain conditions, into an IST integrable system, such as the higher-order NLS (for $\lambda : \mu : \nu = 1 : 6 : 0$) [3, 5], the derivative NLS equation (for $\lambda : \mu : \nu = 0 : 1 : 0$) [7], or the complex modified Korteweg–deVries (KdV) equation (for $\lambda : \mu : \nu = 1 : 6 : 3$) [8].

For the special case $\lambda = \mu = \nu = 0$ the ENLS equation is reduced to the IST integrable conventional NLS equation, which possesses two different types of soliton solutions depending on s (± 1). In the case $s = -1$ (anomalous GVD) the NLS equation has the *bright* soliton solution [9] (satisfying zero boundary conditions), while in the case $s = +1$ (normal GVD), it has the dark soliton solution [10] (satisfying the boundary conditions $|q| \to U$ (U = constant) at $\tau \to \pm\infty$). A limiting case of the latter is the *small-amplitude* dark soliton, closely related to the KdV soliton [11], which is given by

$$q(\tau, \zeta) = U\left(1 - \frac{\delta^2}{2}\,\mathrm{sech}^2\,\Xi\right)\exp[\mathrm{i}U^2\zeta + \mathrm{i}\phi(\tau, \zeta)] \qquad (2)$$

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where the parameter δ ($\delta \ll 1$) is related to the depth of the dark soliton, $\Xi = \delta U(\tau \mp U\zeta \pm \delta^2 U\zeta/2)$ and $\phi(\tau,\zeta) = 2\delta/[1+\exp(2\Xi)]$. Another kind of optical solitary structure, closely connected with the dark soliton, is the so-called *anti-dark* soliton, which exhibits the form of a bright pulse on a continuous wave (cw) background (i.e. it is a dark soliton with reverse sign amplitude). This type of structure has been simultaneously derived by Vekslerchik [12] for $\lambda = \nu = 0$ and, in the small-amplitude limit ($\delta^2 \ll 1$), by Kivshar [13] for $\mu = \nu = 0$.

Unlike the conventional NLS equation (where the value of s determines the type of soliton solution), the ENLS equation has been found to support bright soliton solutions for both the anomalous and normal dispersion regimes ($s = \pm 1$) [7]. Additionally, families of bright and dark solitary wave solutions have been obtained [14] for both regimes, even in the case of pulse propagation at the so-called zero dispersion point (corresponding to zero GVD) [15]. On the other hand, both dark and anti-dark soliton solutions have been obtained in the literature [12, 13], *solely* in the case of the normal dispersion regime ($s = +1$).

As far as the ENLS equation (1) with $s = \pm 1$ is concerned, to our knowledge, the conditions for small-amplitude dark or anti-dark solitary wave formation, as well as the solutions themselves, are not available to date. The present work covers this gap by dealing with the derivation of these solutions, in terms of a KdV, by using an asymptotic perturbation technique. Dark solitary wave solutions are obtained and it is demonstrated that are valid *both* the normal and anomalous dispersion regimes ($s = \pm 1$). This result indicates the possibility of dark solitary wave solutions in wavelength regions not accessible to the conventional form and certain perturbed versions of the NLS, such as the anomalous dispersion regime ($s = -1$). Additionally, anti-dark solitary wave solutions have been obtained in the normal dispersion regime ($s = +1$), and the possibility of transformations from the dark to the anti-dark solitary structure is demonstrated. The dynamics of the derived solitary structures under the so-called stimulated Raman effect [1] have also been studied in the context of a KdV–Burgers (KdV–B) equation. It was found that the solitary structures experience a decrease in their amplitudes and/or their velocities, depending on the direction of propagation and the wavelength region.

The paper is organized as follows. In section 2, the ENLS equation is connected with a KdV equation by means of a perturbation method [11]. The small-amplitude solitary waves are presented in section 3, while section 4 is devoted to the influence of the Raman effect. Finally, in section 5, the main conclusions of this work are reviewed.

2. Connection of the ENLS equation with the KdV equation

In order to derive the small-amplitude solitary structures of equation (1) having the form of dark or anti-dark solitary waves, a perturbative approach similar to the one mentioned in the introduction, will be used. As a first step, one may observe that equation (1) possesses the cw solution $q_{\mathrm{cw}}(\tau,\zeta) = U\exp(\mathrm{i}U^2\zeta)$ ($U =$ constant). Then, a solution of the following form is considered,

$$q(\tau,\zeta) = [U + u(\tau,\zeta)]\exp[\mathrm{i}U^2\zeta + \mathrm{i}\phi(\tau,\zeta)] \tag{3}$$

where $u(\tau,\zeta)$ and $\phi(\tau,\zeta)$ are unknown real functions to be determined. It is readily seen that $u(\tau,\zeta)$ and $\phi(\tau,\zeta)$ express an amplitude and a phase modulation of the cw background, respectively. Apparently, before proceeding with the determination of these functions, it is important to consider the modulational stability of $q_{\mathrm{cw}'}$ because results obtained for an unstable background do not have any physical purport.

The modulational instability in equation (1) has been investigated in [16] for the anomalous dispersion case ($s = -1$) and it has been found that q_{cw} is stable for $U^2(\mu+\nu)^2 - 1 \geqslant 0$. In our case, we can easily verify that this result is generalized to also include the normal dispersion case ($s = -1$), by the inequality $U^2(\mu+\nu)^2 + s \geqslant 0$. This shows that physically acceptable solutions may also be derived for every value of $U(\mu+\nu)$ in the normal dispersion regime.

Upon substituting equation (3) into equation (1), the imaginary part of the resulting equation reads

$$\frac{\partial u}{\partial \zeta} - \frac{s}{2}(U+u)\frac{\partial^2 \phi}{\partial \tau^2} - s\frac{\partial u}{\partial \tau}\frac{\partial \phi}{\partial \tau} - \lambda\frac{\partial^3 u}{\partial \tau^3} + 3\lambda(U+u)\frac{\partial \phi}{\partial \tau}\frac{\partial^2 \phi}{\partial \tau^2} + 3\lambda\frac{\partial u}{\partial \tau}\left(\frac{\partial \phi}{\partial \tau}\right)^2 + U(3\mu+2\nu)(U+2u)\frac{\partial u}{\partial \tau} + (3\mu+2\nu)u^2\frac{\partial u}{\partial \tau} = 0 \qquad (4)$$

while the real part of the resulting equation reads

$$-(U+u)\frac{\partial \phi}{\partial \zeta} - \frac{s}{2}\frac{\partial^2 u}{\partial \tau^2} + \frac{s}{2}(U+u)\left(\frac{\partial \phi}{\partial \tau}\right)^2 + 2U^2u + 3Uu^2 + u^3 + \lambda(U+u)\frac{\partial^3 \phi}{\partial \tau^3} + 3\lambda\left(\frac{\partial^2 u}{\partial \tau^2}\frac{\partial \phi}{\partial \tau} + \frac{\partial u}{\partial \tau}\frac{\partial^2 \phi}{\partial \tau^2}\right) - \lambda(U+u)\left(\frac{\partial \phi}{\partial \tau}\right)^3 - \mu(U^3+u^3)\frac{\partial \phi}{\partial \tau} - 3\mu Uu(U+u)\frac{\partial \phi}{\partial \tau} = 0. \qquad (5)$$

Then, introducing an arbitrary small parameter ε ($0 < \varepsilon \ll 1$), the asymptotic expansions $u = \sum_{n=1}^{\infty} \varepsilon^{2n} u_{n-1}(T, Z)$ and $\phi = \sum_{n=1}^{\infty} \varepsilon^{2n-1} u_{n-1}(T, Z)$ for the unknown functions $u(\tau, \zeta)$ and $\phi(\tau, \zeta)$ are assumed, where $T = \varepsilon(\tau - \Lambda\zeta)$, $Z = \varepsilon^3\zeta$ and Λ is a parameter representing the inverse of the wave velocity in the τ–ζ reference frame (to be determined). The leading order parts for equations (4) and (5), respectively, are

$$\mathrm{O}(\varepsilon^3) : [\Lambda - (3\mu+2\nu)U^2]\frac{\partial u_0}{\partial T} + \frac{sU}{2}\frac{\partial^2 \phi_0}{\partial T^2} = 0 \qquad \mathrm{O}(\varepsilon^2) : 2Uu_0 + (\Lambda - \mu U^2)\frac{\partial \phi_0}{\partial T} = 0. \qquad (6)$$

Assuming that $\Lambda - \mu U^2 \neq 0$, equations (6) lead to the determination of the unknown parameter Λ, which may have two distinct values Λ_+ and Λ_-, given by the following equation:

$$\Lambda_{\pm} = U[U(2\mu+\nu) \pm (U^2(\mu+\nu)^2 + s)^{1/2}]. \qquad (7)$$

Thus, in the linear limit, the wave excitations of the cw background may propagate with two different velocities. Note that in the anomalous dispersion regime ($s = -1$), the inequality $U(\mu+\nu) \geqslant 1$ must hold for $\Lambda_{\pm}$ to be real. This condition verifies the modulational stability of the background. In this regime, both Λ_+ and Λ_- are positive. In the normal dispersion regime ($s = +1$) on the other hand, the velocity Λ_+ is always positive, while the velocity Λ_- can take both positive (for $U(\mu(3\mu+2\nu))^{1/2} > 1$) and negative (for $U(\mu(3\mu+2\nu))^{1/2} < 1$) values. This means that propagation in opposite directions is possible, depending on the amplitude U of the cw background and/or the coefficients μ and ν of the higher-order nonlinear terms.

Proceeding to the next order, namely to order $\mathrm{O}(\varepsilon^5)$, equation (4) reads

$$\frac{\partial u_0}{\partial Z} - s\frac{\partial u_0}{\partial T}\frac{\partial \phi_0}{\partial T} - \frac{s}{2}u_0\frac{\partial^2 \phi_0}{\partial T^2} - \lambda\frac{\partial^3 u_0}{\partial T^3} + 3\lambda U\frac{\partial \phi_0}{\partial T}\frac{\partial^2 \phi_0}{\partial T^2} + 2U(3\mu+2\nu)u_0\frac{\partial u_0}{\partial T} + [(3\mu+2\nu)U^2 - \Lambda]\frac{\partial u_1}{\partial T} - \frac{s}{2}u_0\frac{\partial^2 \phi_1}{\partial T^2} = 0 \qquad (8)$$

while, to order $O(\varepsilon^4)$, equation (4) reads

$$-U\frac{\partial\phi_0}{\partial Z}-\frac{s}{2}\frac{\partial^2 u_0}{\partial T^2}+\frac{s}{2}U\left(\frac{\partial\phi_0}{\partial T}\right)^2+\lambda U\frac{\partial^3\phi_0}{\partial T^3}-(3\mu U^2-\Lambda)u_0\frac{\partial\phi_0}{\partial T}+3Uu_0^2$$
$$+2U^2u_1+(\Lambda-\mu U^2)u_0\frac{\partial\phi_1}{\partial T}=0. \quad (9)$$

Now using equations (6), the temporal derivatives of ϕ_0 can be expressed in terms of u_0. Then, substitution of the resulting expressions into equations (8) and (9), after some algebra, leads to the following KdV equation for the amplitude u_0,

$$c_0\frac{\partial u_0}{\partial Z}-c_1u_0\frac{\partial u_0}{\partial T}+c_2\frac{\partial^3 u_0}{\partial T^3}=0 \quad (10)$$

where

$$c_0=U^2+s(\Lambda-\mu U^2)^2 \quad (11)$$

$$c_1=-2(\Lambda-\mu U^2)[3U+s(3\mu+2\nu)(\Lambda-\mu U^2)]$$
$$+2U\left(\Lambda-3\mu U^2-6s\lambda U^2-\frac{sU^2}{\Lambda-\mu U^2}\right) \quad (12)$$

and

$$c_2=-[s(\Lambda-\mu U^2)(\tfrac{1}{4}+\lambda(\Lambda-\mu U^2))+\lambda U^2]. \quad (13)$$

In this way, the ENLS equation (1), has been connected with the KdV equation (10). Thus, the soliton solutions of equation (10), which will be presented in the following section, are the small-amplitude solitary structures for the ENLS equation. As will be seen, these structures have the form of dark or anti-dark solitary wave solutions. It should be noticed that the term 'solitary wave' rather than 'soliton' solution is used, due to the fact that the ENLS equation is not in general IST integrable, as mentioned in the introduction.

3. Small-amplitude dark or anti-dark solitary waves

The small-amplitude solitary structures for equation (1) can directly be obtained upon using equation (10). Assuming that $c_0c_1c_2\neq 0$, it is readily seen that these solutions have the following form,

$$u_0(T,Z)=-\frac{12c_2}{c_1}k^2\,\mathrm{sech}^2\,\theta \qquad \theta=k(T-\xi(Z)) \quad (14)$$

where k is the amplitude of the conventional KdV soliton [6], $\xi(Z)$ is given by

$$\xi(Z)=\frac{4c_2}{c_0}k^2Z+\xi_0 \quad (15)$$

and ξ_0 is a constant depending on the initial conditions. Notice that a direct comparison of equation (14) and the asymptotic expansion of u with equation (2) leads to a connection between the formal perturbative parameter ε and the parameters of the derived solitary structures, expressed by the simple expression $(24c_2/c_1)k^2\varepsilon^2=\delta^2U$. In addition, using the second of equations (6), the unknown function $\phi_0(T,Z)$ can directly be obtained by means of equation (14):

$$\phi_0(T,Z)=\frac{24Uc_2}{c_1(\Lambda-\mu U^2)}k\tanh\theta. \quad (16)$$

A closer look at equation (14) shows that propagation of different kinds of solitary structures is possible. Thus, equation (14) describes both the case of dark pulse solutions (for $c_2/c_1 > 0$) and the case of anti-dark pulse solutions (for $c_2/c_1 < 0$). On the other hand, equation (15) shows that the direction of propagation of either the dark or the anti-dark pulse depends on the sign of the c_2/c_0. Thus, equation (14) demonstrates both cases of propagation to the right (for $c_2/c_0 > 0$) and propagation to the left (for $c_2/c_0 < 0$).

The derived small-amplitude structures, i.e. the dark and anti-dark solitary wave solutions in equation (14), form a new set of approximate solitary wave solutions of the ENLS equation, which can be supported in *both* the normal ($s = +1$) and the anomalous ($s = -1$) dispersion regimes. As far as the latter regime is concerned, the aforementioned result is in sharp contrast with the conventional form (or certain perturbed versions [11–13]) of the NLS equation, where dark soliton (or solitary wave) solutions hold solely in the normal dispersion regime. It is also noted that since the derived solutions are simultaneously amplitude- and phase-modulated, they cannot be considered as a small-amplitude special case of other solitary solutions of the ENLS equation derived elsewhere (e.g. [14]).

Let us proceed now with the investigation of dark or anti-dark solitary wave formation, which depends on the signs of the coefficients c_0, c_1, c_2, and apply the results to the problem of femtosecond pulse propagation in nonlinear optical fibres. Inspection of equations (11)–(13) shows that these signs depend on the values of the fibre parameters λ, μ and ν, the amplitude U of the cw background and the parameter Λ, which takes two distinct values $\Lambda_\pm$, as shown in equation (7), for either $s = +1$ or $s = -1$. It is evident that the possible changes of the values of these signs may arise from either the change of the fibre parameters (for example due to axial inhomogeneity), or the change of the amplitude of the cw background. However, the available (commercial) fibres are actually axially uniform, since they exhibit a variation of the core radius of order 1% and a negligible variation of the refractive index [1]. On the other hand, the present application deals with pulse propagation in the femtosecond time scale, where the values of the fibre parameters change slightly [4, 7]. Therefore, in order to determine the signs of c_2/c_1 and c_2/c_0, the fibre parameters have been kept fixed at the typical values $\lambda = 0.2$, $\mu = 0.8$ and $\nu = 0.25$, which refer to graded-index monomode fibres and correspond to an initial pulse width $t_0 \approx 50$ fs [4]. On the other hand, the quantity $U(\mu + \nu)$ has been considered as a variable lying in the interval (0, 10) for $s = +1$, or in the interval (1, 10) for $s = -1$ (note that in this case $U(\mu + \nu) > 1$ for Λ to be real). In this way, the 'switching' of the signs of c_2/c_1 and c_2/c_0 for a varying $U(\mu + \nu)$ has been studied and the results, which are divided in two subcases (anomalous and normal dispersion regimes), are as follows.

3.1. Anomalous dispersion regime ($s = -1$)

When the velocity Λ is $\Lambda = \Lambda_+$, the coefficients c_0 c_1 and c_2 do not change sign, namely they are $c_0 < 0$, $c_1 > 0$, $c_2 > 0$. Thus, solution (14) has the form of a dark pulse propagating to the left. On the other hand, when the velocity Λ is $\Lambda = \Lambda_-$, the signs of the coefficients c_1 and c_2 do not change, namely they are $c_1 > 0$ and $c_2 > 0$, but the sign of the coefficient c_0 changes for $U(\mu + \nu) \cong 1.85$. Thus, in this case $c_2/c_1 < 0$, while $c_2/c_0 > 0$ for $U(\mu + \nu) < 1.85$ and $c_2/c_0 < 0$ for $U(\mu + \nu) > 1.85$. Solution (14) describes again a dark solitary wave, which propagates to the right (left) for $U(\mu + \nu) < 1.85$ ($U(\mu + \nu) > 1.85$).

3.2. Normal dispersion regime ($s = +1$)

When the velocity Λ is $\Lambda = \Lambda_+$, the coefficients in equations (11)–(13) do not change sign, namely they are $c_0 > 0$, $c_1 < 0$, $c_2 < 0$. Thus, in this case, solution (14) represents a dark pulse propagating to the left. On the other hand, when the velocity Λ is $\Lambda = \Lambda_-$, the sign of the coefficient c_0 does not change, namely $c_0 > 0$, but the signs of the coefficients c_1 and c_2 change as follows: $c_1 > 0$ for $1.04 < U(\mu + \nu) < 1.33$ and $c_1 < 0$ for $U(\mu + \nu) < 1.04$ or $U(\mu + \nu) > 1.33$, while $c_2 > 0$ for $U(\mu + \nu) < 0.65$ and $c_2 < 0$ for $U(\mu + \nu) > 0.65$. Thus, equation (14) represents an anti-dark pulse propagating to the right for $0 < U(\mu + \nu) < 0.65$, a dark pulse propagating to the left for $0.65 < U(\mu + \nu) < 1.04$, an anti-dark pulse propagating to the left for $1.04 < U(\mu + \nu) < 1.33$ and finally a dark pulse propagating to the left for $1.33 < U(\mu + \nu) < 10$.

It is important to mention that at the two critical points $U(\mu + \nu) = 1.04$ and 1.33 where the coefficient c_1 of the nonlinear term of the KdV equation (10) vanishes, small-amplitude dark-to-anti-dark and anti-dark-to-dark solitary wave transformations occur, respectively. On the other hand, it should be mentioned that these transformations do not occur at the critical point $U(\mu + \nu) = 0.65$ where the linear dispersion coefficient c_2 of equation (10) vanishes. This fact arises from the analysis of a KdV equation with coefficients exhibiting changes of their signs, which has been performed in the past by using both numerical [17] and analytical [18] methods. It is also worth noticing that the above-mentioned results do not significantly depend on higher-order nonlinear terms, e.g. $u_0^2(\partial u_0/\partial T)$, which may additionally be taken into account [18].

In conclusion, small-amplitude dark solitary waves can be formed in both the normal and the anomalous dispersion regimes. However, small-amplitude anti-dark solitary wave formation, as well as transformations from one mode to the other, can solely be observed in the normal dispersion regime.

4. Influence of the Raman effect

The dynamics of the solitary structures found in the previous section has been studied in the absence of the Raman effect [1]. This effect gives rise to a frequency down-shift of bright solitons and leads to fission of soliton bound states [3]. On the other hand, the Raman effect leads to temporal self-shift of dark solitons and, consequently, its influence is more destructive [11]. In the framework of the ENLS equation, the Raman effect is effectively described with an additional term on the right-hand side of equation (1), which now takes the following form [2–5],

$$\mathrm{i}\frac{\partial q}{\partial \zeta} - \frac{s}{2}\frac{\partial^2 q}{\partial \tau^2} + q|q|^2 - \mathrm{i}\lambda\frac{\partial^3 q}{\partial \tau^3} + \mathrm{i}\mu\frac{\partial}{\partial \tau}(q|q|^2) + \mathrm{i}\nu q\frac{\partial}{\partial \tau}(|q|^2) = \epsilon q\frac{\partial}{\partial \tau}(|q|^2) \qquad (17)$$

where ϵ is a positive real constant representing the Raman gain parameter. At low frequencies of the small-amplitude perturbations the Raman term $\epsilon q\partial(|q|^2)/\partial\tau$ does not alter significantly the modulational stability of the background [16]. Therefore, we may modify equation (5) by adding on its right-hand side an additional term of the form $2\epsilon(U^2 + u^2 + 2u)(\partial u/\partial\tau)$, which amounts to adding a term $2U^2(\epsilon/\varepsilon)(\partial u_0/\partial\tau)$ on the right-hand side of equation (9). As a result, the first-order evolution equation for the amplitude u_0 takes the form

$$c_0\frac{\partial u_0}{\partial Z} + c_1 u_0\frac{\partial u_0}{\partial T} + c_2\frac{\partial^3 u_0}{\partial T^3} = c_3\frac{\partial^2 u_0}{\partial T^2} \qquad (18)$$

where $c_3 = (\epsilon/\varepsilon)U^2$. Equation (18) has the form of a KdV–B equation. As can be seen, for $c_3 = 0$ equation (18) is reduced to a KdV equation, while for $c_2 = 0$ it is reduced to a Burgers equation. The KdV–B equation does not possess soliton solutions. However, this equation has travelling solutions in the form of shock waves with oscillatory tails [19]. Notice that similar shock waves were found in the context of nonlinear optics [20] in terms of a NLS equation with $s = +1$, driven by the Raman term. Nevertheless, these kinds of solutions are not connected with the problem in hand, since our purpose is to investigate the influence of the Raman effect on the derived solitary structures.

For the aforementioned reasons, we restrict ourselves to the case $c_3/c_2 \ll 1$, in order to study the dynamics of solutions (14) in the presence of the Raman term ($\epsilon \neq 0$) by means of the perturbation theory of solitons [21, 22]. According to this approach, the solution of equation (18) is expressed as

$$u_0(T, Z) = u_s(\theta, k) + \Delta u(\theta, Z) \tag{19}$$

where $u_s(\theta, k)$ has the form of the solitary pulse described by equation (14) and $\Delta u(\theta, Z)$ is the correction to the solitary pulse in the adiabatic approximation. As far as the amplitude k and the velocity $\mathrm{d}\xi/\mathrm{d}Z$ of the solitary part of the solution are concerned, they are assumed to be not constants, but slowly-varying functions of Z, i.e.

$$u_s(\theta, k) = -\frac{12c_2}{c_1} k^2(Z)\,\mathrm{sech}^2\,\theta \qquad \theta = k(Z)[T - \xi(Z)]. \tag{20}$$

The evolution of these parameters is described according to the following equations,

$$k(Z) = k(0)(1 + Z/Z_0)^{-1/2} \tag{21}$$

where

$$Z_0 = \frac{c_0}{c_3} \frac{15}{16k^2(0)} \tag{22}$$

and

$$\frac{\mathrm{d}\xi}{\mathrm{d}Z} = \frac{4c_2}{c_0} k^2(Z) + \frac{8c_3}{15c_0} k(Z). \tag{23}$$

On the other hand, the deformation $\Delta u(\theta, Z)$ is transformed into a flat tail at a few solitary lengths behind the solitary structure. The asymptotics of $\Delta u(\theta, Z)$ are given by

$$\Delta u(\theta, Z) \approx \frac{16c_3}{5c_1} k^2(Z)\theta \exp(-2\theta) \qquad \theta \gg 1 \tag{24}$$

and

$$\Delta u(\theta, Z) \approx -\frac{16c_3}{5c_1} k(Z)[1 + \theta^2 \exp(2\theta)] \qquad \theta \ll -1. \tag{25}$$

Equations (21)–(23) show that the evolution of the parameters k and $\mathrm{d}\xi/\mathrm{d}Z$ of the solitary structures (14) depends on the signs of the coefficients c_0 and c_2 (note that c_3 is always positive). This means that the aforementioned evolution is closely connected to the direction of the motion of the solitary structure. According to the relevant investigation presented in the previous section, the behaviour of the small-amplitude solitary structures is described as follows.

In the anomalous dispersion case ($s = -1$), the dark solitary waves propagating to the left ($c_2/c_0 < 0$) increase their amplitude (since $c_0 < 0$) and decrease their velocity. However, the dark solitary wave propagating to the right ($c_2/c_0 > 0$) decreases both its amplitude (since $c_0 > 0$) and its velocity. This structure is expected to disappear at distances of order Z_0, where Z_0 is given by equation (22). As far as the normal dispersion case ($s = +1$) is concerned, both the dark and the anti-dark solitary waves propagating to the left ($c_2/c_0 < 0$) decrease their amplitudes ($c_0 > 0$) and increase their velocity. On the other hand, the anti-dark solitary wave propagating to the right ($c_2/c_0 > 0$) decreases both its amplitude ($c_0 > 0$) and its velocity. These structures are also expected to disappear at distances of order Z_0.

In conclusion, under the influence of the Raman effect, the solitary structures propagating to the right experience a decrease in both their amplitudes and their velocities. On the other hand, the evolution of the solitary structures propagating to the left depends on the wavelength region: in the anomalous dispersion regime, where $s = -1$, they increase their amplitudes and decrease their velocities, while in the normal dispersion regime, where $s = +1$, they decrease their amplitudes and increase their velocities.

5. Conclusions

The problem of small-amplitude solitary structure formation for an extended nonlinear Schrödinger equation has been considered. This model is suitable for describing propagation of femtosecond duration pulses in nonlinear optical fibres. Upon using a perturbation technique, the aforementioned equation has been connected with the Korteweg–deVries equation and small-amplitude dark solitary wave solutions have been derived for both the normal and anomalous dispersion regimes.

This result is in sharp contrast with the conventional form and certain perturbed versions of the nonlinear Schrödinger equation, where dark soliton or dark solitary wave solutions hold solely in the normal dispersion regime. In addition, in this latter regime, small-amplitude anti-dark solitary wave solutions have been obtained and the possibility of transformations from the dark to the anti-dark solitary structure has been demonstrated. The results obtained in this work generalize the results of previous important theoretical works (e.g. [11–13]). This is due to the fact that all the physically significant higher-order effects have been simultaneously taken into account in the nonlinear Schrödinger equation.

Finally, the influence of the Raman effect on the derived small-amplitude solitary structures has also been investigated. It has been found that the dynamics of these structures is governed by a KdV–B equation and, under certain conditions, their behaviour has been studied by means of the perturbation theory for solitons. The results that have been obtained show that the solitary structures experience a decrease in their amplitudes and/or their velocities, depending on the direction of propagation and the wavelength region.

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